ON THE DEVELOPMENT OF A THEORY OF IDEAL PLASTICITY

(K POSTROENIIU TEORII IDEAL'NOI PLASTICHNOSTI)

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In this work some extremal properties of an ideal plastic flow satisfying Tresca's plasticity condition are considered. These properties distinguish Tresca's plasticity condition from a class of admissible plasticity conditions defined below.

The plastic state of a body is determined by its residual strains. This concept presupposes an unloading process, and since the occurrence of a plastic state is connected with the characteristic deviation of a stress-strain diagram from linearity, the analysis of plastic properties in its simplest manifestations is contained within an analysis of the mechanical changes of materials during the loading process.

Plastic flow at a point of a rigid-plastic body occurs when a certain stress combination at this point reaches its limiting value. Hence, the plasticity condition can be written as

$$f(\sigma_1, \sigma_2, \sigma_3) = \text{const} \tag{1}$$

where σ_1, σ_2, σ_3 are principal stresses.

Possible simplifications of plasticity condition are achieved to a considerable degree on the assumption of homogeneity of the material and of an ideal and isotropic character of plastic flow.

The first assumption, in its simplest form, states that plastic flow is independent of a series of parameters which characterize changes in initial properties of a material.

The second states that the material is non-strain-hardening, and thus (1) is independent of some parameters which characterize some changes of the material during the process of plastic flow.

The third assumption should be divided into two parts: one concerning the initial isotropy of the body, and the other concerning the absence of acquired anisotropy. These two properties are, essentially, independent. Indeed, one may assume that an initially isotropic body becomes anisotropic during the process of plastic deformation (this is supported by all experimental results), or that an initially anisotropic body retains its anisotropic character during the plastic flow. (This is analogous, for instance, to the theory of elastic anisotropic bodies.) The greatest simplification is evidently achieved on the assumption of the absence of any kind of anisotropy. Such an assumption is made in the theory of plasticity, which limits itself to the study of the influence of changes in the mechanical properties of materials during the process of loading.

The ideal and isotropic character of plastic flow implies that the function f in (1) retains its form during the whole process of plastic flow. Moreover, the isotropy conditions require that condition (1) must be an invariant under some class of transformations of the arguments, in order to secure the equivalent role of these arguments.

The assumption of an ideal and isotropic character of plastic flow of metals is in direct contradiction to the evidence of experimental investigations. These investigations invariably indicate that metals become anisotropic and that a considerable deviation from an ideal character of deformations exists during the plastic flow. In constructing the mathematical theory of plasticity, these properties of real materials are incorporated in subsequent generalisations of the theory which do not allow for these phenomena.

The fact that the plastic properties of materials are independent of hydrostatic pressure means that in the space of principal stresses σ_1, σ_2, σ_3, condition (1) is geometrically represented by a cylinder whose generators are parallel to the line $\sigma_1 = \sigma_2 = \sigma_3$. To represent the plasticity condition it thus is sufficient to consider a curve which is a cross-section of the cylinder representing the yield surface with the plane $\sigma_1 + \sigma_2 + \sigma_3 = 0$. The curve so obtained is called the yield locus and is shown in Fig. 1.

The simplest necessary properties of a yield locus are: first, it cannot pass through the origin; and secondly, any radius drawn from the origin cuts it once and once only. The necessary character of these conditions is obvious and requires no further explanation.

The isotropic properties of materials require that the yield locus must be symmetrical with respect to axes σ_1, σ_2, σ_3 (Fig. 1).

A considerable simplification is obtained by assuming that (1) is independent of the change of stress sign. The mechanical meaning of this fact is that the material behaves similarly in tension and in compression. The stress sign's being independent of the plasticity condition obviously leads to the conclusion that the yield locus is symmetrical with respect to axes perpendicular to the σ_1, σ_2 and σ_3 axes. Hence the yield locus consists of twelve similar arcs.

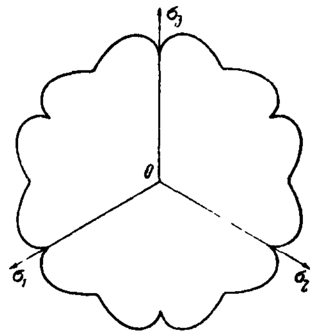

Fig. 1.

Thus in establishing a theory of plasticity considering characteristic changes in the mechanical properties of a material, the following assumptions are made:

(1) absence of elastic strains in the body (rigid-plastic behavior);

(2) homogeneity of properties of the material;

(3) absence of strain-hardening (ideal flow character);

(4) absence of initial and acquired anisotropy;

(5) absence of the influence of hydrostatic pressure on plastic properties of the material;

(6) absence of differentiation between tensile and compressive properties of the material.

Here such factors as influence of temperature, inertia and other body forces, etc, are not mentioned. In short, all these assumptions so familiar in the theory of elasticity, are absent.

The assumptions (2)-(6) permit substantial simplification of the yield condition (1), relaxation of any of the assumptions (1)-(6) leading to a generalization of the theory of plasticity under consideration.

An indeterminate form of the yield locus, and also of the stress-plastic-strain relationship, obviously offers considerable possibilities for developing different plasticity theories. It should be clearly borne in mind, however, that within a definite set of assumptions regarding the idealized properties of materials, the processes described must possess properties of complete definiteness and uniqueness, and be characterized by extremal properties in comparison to all possible processes.

One of the fundamental steps in the establishment of any theory of plasticity was made by Mises, who determined extremal properties of plastic flow for a plasticity condition represented in the form of a plastic potential. The plastic potential is understood to be a function of the stresses $g(\sigma_{ij})$ which determined a relationship between the components of plastic strains. In fact it is possible to establish extremal properties of plastic flow only for the simplest case of this relationship, namely $g \equiv f$.

Mises' theory asserts that for the relationship

$$d\varepsilon_{ij} = d\lambda\,(\partial f\,(\sigma_{ij}) / \partial\sigma_{ij}) \tag{2}$$

where ϵ_{ij} are components of plastic strain, the work of the stresses on the corresponding strain increments attains its maximum.

If we employ vector notation, then (2) means that the direction of the plastic strain increment vector coincides with the direction of a normal to the plasticity surface. The law for plastic flow, which is determined by (2), is called an "associated" plastic flow rule.

Koiter's generalization of Mises' results is known as the theory of generalized plastic potential According to this theory the corner points of the plasticity surface are interpreted as limiting cases of a smooth surface. At the corner points the plastic-strain increment vector may thus take any arbitrary direction, provided that it is contained within the normal directions to the plasticity surface at the points lying arbitrarily close to the corner points.

The local character of the extremal properties determined by the Mises theory should be pointed out.

Investigations conducted by Hill, Prager, Koiter, Drucker and others have demonstrated that the theory of the plastic potential, including the generalized potential, permits a formulation of the uniqueness theorems and the establishment of an integral variational principle. These investigations also show that the theory of the plastic potential appears to be the only one acceptable for establishing the said properties of uniqueness and extremum of plastic flow, and it consequently appears to be a necessary logical link in developing the simplest theory of plasticity, for it follows that the yield locus has to be a convex curve, since otherwise the theory means that some solutions are indeterminate. The yield locus is said to be convex if it always lies on one side of the tangent at any arbitrary point. At the corner points the tangent may take any arbitrary direction between the left-hand and right-hand tangents at such a point. An analytical condition of convexity of a differentiable function (1) is

$$f(\sigma_{ij}') - \sigma_{ij}' \frac{\partial f}{\partial \sigma_{ij}} \geq f(\sigma_{ij}) - \sigma_{ij} \frac{\partial f}{\partial \sigma_{ij}} \tag{3}$$

The relationships (2) and (3) and their generalization to the stepwise smooth surface permit the following to be established: if on one part of the surface of a rigid-ideally plastic body surface tractions, and on the other part displacement increments are prescribed, then the work of prescribed tractions along the corresponding displacement increments attains its minimum for a real strain increment field in comparison to all other kinematically admissible strain increment fields. It can also be established that the work of the surface tractions along prescribed displacement increments is maximum for a real stress field in comparison to all statically admissible stress fields.

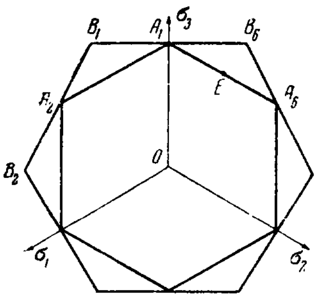

Fig. 2.

Note that a kinematically admissible strain increment field is determined by a displacement increment field which satisfies prescribed boundary conditions and in which the kinematically admissible displacement increment discontinuities are possible. A statically admissible stress field is one which satisfies prescribed equilibrium equations and the plasticity condition, and in which statically admissible stress discontinuities are possible.

According to the uniqueness theorems, boundary conditions determine a unique plastic state, etc.

Theoretical considerations permit the construction of two polygons $A_1 \ldots A_6$ and $B_1 \ldots B_6$ such that all other symmetrical yield loci lie within these two polygons (Fig. 2). Nevertheless, however, there exist an infinite number of yield loci permitting the development of a theory of rigid-ideally -plastic bodies satisfying the said considerations of uniqueness, determinacy and extremum of the processes described.

The problem of determining a yield condition is here treated as a variational problem of selecting the process of plastic flow possessing certain extremum properties in comparison with all other processes determined by admissible plasticity conditions. An admissible plasticity condition is understood to be a condition with a convex symmetrical yield locus lying between the hexagons in Fig. 2.

It is easy to see that the considerations on distinguishing a true plasticity condition from all admissible ones are at bottom analogous to the Mises considerations on determinating a true law of plastic flow from all the possible laws of such a flow.

We shall now establish a local theorem permitting the determination of a plasticity condition which is the required solution to the problem. Let us then accept that a true plasticity condition differs from all possible ones by the fact that its contribution to the work of the stresses along prescribed strain increments is a minimum. Let $d\epsilon_1$, $d\epsilon_2$, $d\epsilon_3$ be the prescribed strain increments. The work done by the stresses is

$$dW = \sigma_1 d\varepsilon_1 + \sigma_2 d\varepsilon_2 + \sigma_3 d\varepsilon_3 \tag{4}$$

or in vector notation

$$dW = |\mathbf{d\varepsilon}| \, |\boldsymbol{\sigma}| \cos \varphi$$

where ϕ is the angle between $d\epsilon$ and σ. The magnitude $|d\epsilon|$ is given, and we have to examine only the magnitude $|\sigma| \cos \phi$. Clearly, if a strain increment vector has direction OC, noncoincident with OA_i, (Fig. 3), then, dropping from A_i a perpendicular OD to OC, we obtain that, among all possible yield loci for a prescribed plastic strain increments vector whose direction coincides with OC, the minimum value of the work is determined by an expression

$$dW = |\mathbf{d\varepsilon}| \, |\sigma| \qquad (|\boldsymbol{\sigma}| = OD)$$

Obviously, $OD_1 \cos \phi = OD$ (Fig. 3).

The minimum work done by the stresses on prescribed strain increments is represented by a segment OE, except possibly for the points A_i. Consequently, the yield locus required is a hexagon $A_1 A_2 \ldots A_6$, which represents the familiar Tresca plasticity condition,

$$\max\{|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3, |\sigma_3 - \sigma_1|\} - 2k = 0 \qquad (k = \mathrm{const})$$

The equation of an arbitrary line A_i, A_{i+1} can be written as $\sigma_i - \sigma_j = 2k$. Hence it follows that

$$d\varepsilon_k = 0, \ d\varepsilon_i + d\varepsilon_j = 0$$

After some computations we get

$$dW = 2kd\varepsilon_i = -2kd\varepsilon_j.$$

Now consider points A_i, where

$$\sigma_i - 2k = \sigma_j = \sigma_k.$$

It follows from this that $dW = 2kd\epsilon_i$. Moreover, this expression is independent of the curvature of the yield locus at point A_i. This concludes the proof of the 'local' theorem, which establishes the extremal properties of the Tresca plasticity condition in comparison with any other possible plasticity condition.

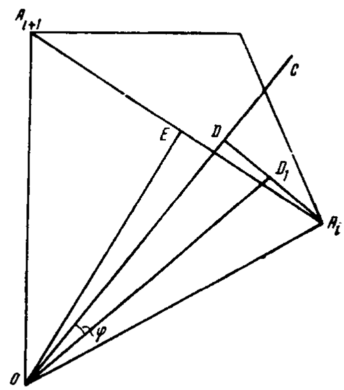

Fig. 3.

Now consider a theorem which establishes integral extremal properties of Tresca's plasticity condition. We will show that in an rigid-ideally-plastic body the effect of the surface forces X_i on prescribed displacement increments $du_i{}^*$ is a minimum for the Tresca plasticity condition. Then let the stresses $\sigma_i{}'$, the strains $\epsilon_i{}'$ and the surface forces $X_i{}'$ correspond to some admissible plasticity condition, and the stresses σ_i, the strains ϵ_i and the surface forces X_i correspond to the Tresca condition. It is necessary to show the validity of

$$\int_s X_i{}' du_i{}^* ds - \int_s X_i du_i{}^* ds \geqslant 0 \tag{5}$$

Employing volume integrals, we can show that inequality (5) is equivalent to the inequality

$$\int_v \sigma_i{}' d\varepsilon_i{}' dv - \int_v \sigma_i d\varepsilon_i dv \geqslant 0 \tag{6}$$

Some modifications of (6) give

$$\int_v \sigma_i' d\varepsilon_i' dv - \int_v \sigma_i d\varepsilon_i dv = \int_v |d\boldsymbol{\varepsilon}'| |\boldsymbol{\sigma}'| \cos\varphi' dv - \int_v |d\boldsymbol{\varepsilon}| |\boldsymbol{\sigma}| \cos\varphi dv =$$

$$= \int_v |d\boldsymbol{\varepsilon}'| (|\boldsymbol{\sigma}'| \cos\varphi' - |\boldsymbol{\sigma}| \cos\varphi) dv + \int_v |d\boldsymbol{\varepsilon}'| |\boldsymbol{\sigma}| \cos\varphi dv - \int_v |d\boldsymbol{\varepsilon}| |\boldsymbol{\sigma}| \cos\varphi dv > 0 \qquad (7)$$

The inequality (7) is valid, since from the 'local' theorem it follows that $|\sigma'| \cos \phi' > |\sigma| \cos \phi$. Moreover, the inequality

$$\int_v |d\boldsymbol{\varepsilon}'| |\boldsymbol{\sigma}| \cos\varphi dv > \int_v |d\boldsymbol{\varepsilon}| |\boldsymbol{\sigma}| \cos\varphi dv$$

is also valid, since the strain increment field $d\epsilon_i'$ is kinematically admissible in relationship to the stress state σ_i corresponding to the Tresca condition.

A theorem asserting that the work of prescribed surface tractions is a minimum for the Tresca condition could be proved analogously.

The theorems discussed above permit various generalizations, including a consideration of rigid regions, strain-hardening, etc.

In the theory of rigid-ideally-plastic bodies two essential conditions are used, namely the Tresca and the Mises. Numerous investigations demonstrated that the Mises condition is in closer agreement with experimental evidence than the Tresca.

Without questioning the validity of the experimental results, we may state that the closer agreement of the Mises condition than the Tresca with the experimental evidence is to be explained by the influence of secondary factors bearing no relationship to the theory of rigid-ideally-plastic materials, but related to anisotropy, strain hardening, etc. Therefore, even if the theory of rigid-ideally-plastic materials based on the Mises plasticity condition were in general agreement with the evidence in practice, the agreement would be due to the fact that the Mises plasticity condition modifies the Tresca condition so that to some extent it indirectly allows for factors of nonideal aspects of real materials. However, any modification of the Tresca plasticity condition could at best claim some practical advantages and no theoretical significance at all. Incidentally, this is what Mises himself had in mind when he proposed his plasticity condition as a convenient mathematical approximation to the Tresca condition. In areas of plasticity where it was possible to achieve some definite success, however, as in the theory of torsion and plane strain, the Mises and Tresca conditions essentially coincide.

In three-dimensional problems, the Mises plasticity condition led to insurmountable difficulties. On the other hand, it has been shown [11] that the application of the Tresca condition and the associated law of plastic flow leads to statically determinate problems, and also permits the application, with appropriate modifications, of the whole mathematical apparatus developed in the theory of torsion and plane problems.

We will make a few remarks on the development of the static theory of free-flowing [granular, pulverulent] media. The properties of free-flowing media, in the simplest case, are determined by the relationship

$$f(\sigma_1, \sigma_2, \sigma_3) = \varphi(\sigma_n) \tag{8}$$

where σ_n is the normal pressure. In the limiting case, for $\phi(\sigma_n)$ = const, we have $f(\sigma_1, \sigma_2, \sigma_3) = \tau_{max}$, which is exactly the Tresca condition.

Theoretical considerations should obviously lead to a rigid-free-flowing medium whose motion is determined by (8). Relationship (8) could be considered as a "free-flowing" potential. Moreover, it is clear that the relationship (8) must have the form

$$\tau_{max} = \varphi(\sigma_n) \tag{9}$$

Any different theory will be determined by some definite form of the function $\phi(\sigma_n)$. Obviously, an essential object is to prove the static determinacy of a general problem described by relationship (9), and when a full limiting state exists, corresponding to the maximum freedom of motion of a free-flowing body [11].

The most important problem of any further development of the theory of plasticity is the problem of a reasonable determination of strain-hardening, which is obviously connected with the strains and, in principle at least, can be determined as an invariant which depends in a continuous manner on the strains or their increments, etc). Strain-hardening can possibly be determined by the amount of plastic work, by the maximum or octahedral shear, etc.

Evidently extremal theorems must exist which determine a true measure of strain-hardening and which single it out from the class of all possible measures.

It is quite possible that at this new stage of development, no theory of plasticity can be conceived without bringing in thermodynamic relationships. Thermodynamic aspects do not appear in the theory of ideally plastic materials or in the theory of incompressible fluids in hydrodynamics.

At a certain stage of development in the theory of plasticity the so-called classical approach is incapable of supplying a description of the investigated processes. This situation may compel us to approach the problem from the point of view of the microstructure of deformations.

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